The Principle of Relativity, Kinematics and Algebraic Relations

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Based on the principle of relativity and the postulate on universal invariant constants (c, l), all possible kinematics can be set up with sub-symmetries of the Umov-Weyl-Fock transformations for the inertial motions. Further, in the combinatory approach, all these symmetries are intrinsically related to each other, e.g. to the very important dS kinematics for the cosmic scale physics.

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I. INTRODUCTION

In order to face the challenges of precise cosmology [1, 2], it is needed to re-examine the principles of Einstein's theory of relativity from the very beginning [3]. Especially, whether the principle of relativity should be generalized to the de Sitter (dS) spacetime, to which our universe is possibly accelerated expanding asymptotically.

This is definitely the case and there are two other kinds of special relativity in the dS/anti-dS (AdS) spacetimes of radius l [4–17] based on the principle of relativity and the postulate on universal constants: *There are two universal invariant constants: the speed of light c and the length l.* If the radius l is linked to the cosmological constant Λ from precise cosmology, $l^2 \cong 3/\Lambda$, the dS special relativity provides the new kinematics for the cosmic scale physics [8, 15–17]. Hereafter, the principle with the postulate is simply called the principle of relativity with (c, l) (the $PoR_{c,l}$ for short). Moreover, since the inertial motion equation is invariant under time reversal and space inversion, the $PoR_{c,l}$ connotatively require the invariance under them.

Since the physical space and time coordinates must have right dimensions, the $PoR_{c,l}$ should always make sense. Although l does not explicitly appear in Einstein's special relativity and so do l and c in Newtonian mechanics, these parameters should be there implicitly. Actually, all possible kinematics are deeply related to each other based on the $PoR_{c,l}$.

This has been shown for the Poincaré algebra $\mathfrak{iso}(1,3)$ in Einstein's special relativity in the combinatory approach [3]. It is started with the most general transformations among inertial motions, i.e. the linear fractional transformations with common denominators (LFTs) of twenty-four parameters, called the Umov-Weyl-Fock-transformations $\mathcal{T}$, which form the inertial motion transformation group $IM(1,3)$ with inertial motion algebra $\mathfrak{im}(1,3)$. As its subalgebra, the Poincaré algebra contains the Lorentz algebra $\mathfrak{so}(1,3)$ for spacetime isotropy. Then $IM(1,3)$ is denoted by $IM_L(1,3)$. For the dS/AdS special relativity based on the $PoR_{c,l}$, there are the LFTs with Lorentz isotropy of dS/AdS-group $SO(1,4)/SO(2,3)$ among the inertial motions in the Beltrami model of dS/AdS-spacetime, respectively[8]. So, these LFTs also form subgroups of $IM_L(1,3)$. However, there are thirty parameters for Poincaré, dS and AdS groups in total. This seems a puzzle if no relations among them. Actually, three kinds of special relativity form a special relativity triple with the common Lorentz isotropy [3]. Further, the translations for transitivity in $\mathfrak{so}(1,4), \mathfrak{so}(2,3), \mathfrak{iso}(1,3)$, denoted as $\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{p}$, are related in the space of $\mathfrak{im}_L(1,3)$: The Euclid translations $\mathbf{P}_\mu \in \mathfrak{p}$ can be given by the plus combination of the Beltrami translations $\mathbf{P}^\pm_i \in \mathfrak{d}_\pm$, i.e. $\mathbf{P}_\mu = \frac{1}{2}(\mathbf{P}^+_\mu + \mathbf{P}^-_\mu)$. While the other generators transform them from one to another. Moreover, corresponding groups and spacetimes are also related. Further, the minus combination leads to the pseudo-translations $\mathbf{P}'_\mu = \frac{1}{2}(\mathbf{P}^+_\mu - \mathbf{P}^-_\mu)$ for the second Poincaré algebra $\mathfrak{p}_2$. Thus, in the space of $\mathfrak{im}(1,3)$, the dS/AdS algebras with an

algebraic doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ can be transformed to two Poincaré algebras with a doublet $(\mathfrak{p}, \mathfrak{p}_2)$ and vice versa, while both c and l are unchanged. This shows that the combinatory approach is very different from the contraction approach [21, 22].

In this paper, by means of the combinatory approach we show that if the Lorentz isotropy is relaxed to the space isotropy of rotation algebra $\mathfrak{so}(3)$, all other kinematic algebras of ten generators [22] can also be set up based on the $PoR_{c,l}$, in addition to the relativistic algebras and their geometric counterparts, i.e. three algebras $\mathfrak{so}(5), \mathfrak{so}(4,1), \mathfrak{iso}(4)$ denoted as $\mathfrak{r}, \mathfrak{l}, \mathfrak{e}$ for 4-Riemann/Lobachevsky/Euclid geometry, respectively. And all these symmetries are intrinsically related to each other, e.g. to the dS algebra that is very important for the cosmic scale physics.

As usual, in addition to the space isotropy algebra $\mathfrak{so}(3)$ of generators $\mathbf{J}_i$ without dimension, there are four types of time and space translations: the Beltrami ones and the Euclid ones with their pseudo counterparts: $\{\mathcal{H}\} := \{H^\pm, H, H'\}$ of dimension $[\nu], \nu := c/l$ called the Newton-Hooke constant, and $\{\mathbf{P}\} := \{\mathbf{P}_i^\pm, \mathbf{P}_i, \mathbf{P}'_i\}$ of dimension $[l^{-1}]$, as scalars or vectors of $\mathfrak{so}(3)$, respectively, as well as four types of boosts: $\{\mathbf{K}\} := \{\mathbf{K}_i, \mathbf{N}_i, \mathbf{K}_i^{\mathfrak{g}}, \mathbf{K}_i^{\mathfrak{c}}\}$ of the Lorentz, geometric, Galilei and Carroll boost of dimension $[c^{-1}]$ as vector representations of $\mathfrak{so}(3)$. Namely[1],

$$[\mathbf{J}, \mathbf{J}] = \mathbf{J}, \quad [\mathbf{J}, \mathcal{H}] = 0, \quad [\mathbf{J}, \mathbf{P}] = \mathbf{P}, \quad [\mathbf{J}, \mathbf{K}] = \mathbf{K}. \tag{1}$$

In fact, for the special relativity triple, there is an algebraic quadruplet $(\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{p}, \mathfrak{p}_2)$ made of four triplets and six doublets simplicially. If we replace the Lorentz boost $\mathbf{K}_i$ by the geometric one $\mathbf{N}_i$, the isotropy algebra $\mathfrak{so}(1,3)$ becomes $\mathfrak{so}(4)$. With suitable translations, it follows three algebras $\mathfrak{r}, \mathfrak{l}, \mathfrak{e}$ of Riemann/Lobachevsky/Euclid geometry, respectively, and the corresponding algebraic multiplets. If we require both Euclid translations H and $\mathbf{P}_i$, it follows the Galilei and the Carroll algebra $\mathfrak{g}, \mathfrak{c}$ of a doublet $(\mathfrak{g}, \mathfrak{c})$ with the Galilei boost $\mathbf{K}_i^{\mathfrak{g}}$ and the Carroll boost $\mathbf{K}_i^{\mathfrak{c}}$, respectively. If we replace H by the Beltrami one H^+, H^- and still keep $\mathbf{P}_i$, the Newton-Hooke/anti-Newton-Hooke ($NH_\pm$) algebras $\mathfrak{n}_\pm$ follow with the Galilei boost $\mathbf{K}_i^{\mathfrak{g}}$, respectively. While, if we replace $\mathbf{P}_i$ by the Beltrami ones $\mathbf{P}_i^+, \mathbf{P}_i^-$ and still keep H, the Hooke-Newton/anti-Hooke-Newton ($HN_\pm$) algebras $\mathfrak{h}_\pm$ follow with the Carroll boost $\mathbf{K}_i^{\mathfrak{c}}$, respectively [2]. Further, other algebras can also be given easily by the combinatory approach in the space of $\mathfrak{im}(1,3)$. As the second Poincaré algebra $\mathfrak{p}_2$ and the doublet $(\mathfrak{p}, \mathfrak{p}_2)$, there are always the second algebra for each algebra with the pseudo-translations H' or $\mathbf{P}'_i$ and a doublet for each pair of them, except the dS/AdS algebras $\mathfrak{d}_\pm$ and their geometric counterparts $\mathfrak{r}, \mathfrak{l}$. Moreover, there are also algebraic multiplets made of lower multiplets simplicially with respect to the $\mathfrak{im}(1,3)$, so all these kinematic algebras are intrinsically related. For example, by combining the boosts of the Galilei and Carroll algebras $\mathfrak{g}, \mathfrak{c}$, it follows the Poincaré algebra $\mathfrak{p}$, which leads to the dS algebra $\mathfrak{d}_+$ directly by the combination of the translations and pseudo-translations of $\mathfrak{p}_2$.

This paper is arranged as follows. In section II, we briefly recall the Umov-Weyl-Fock transformations for the $PoR_{c,l}$ and how to get the algebras of three kinds of special relativity, their triple with algebraic relations and their geometric counterparts. In section III, we first get the Galilei and Carroll algebras $\mathfrak{g}, \mathfrak{c}$ as well as their second ones. Then we get the $NH_\pm$ algebras $\mathfrak{n}_\pm$, the $HN_\pm$

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[1] Hereafter, e.g. $[\mathbf{J}, \mathbf{P}] = \mathbf{P}$ is a shorthand of $[\mathbf{J}_i, \mathbf{P}_j^\pm] = -\epsilon_{ij}{}^k \mathbf{P}_k^\pm$ etc. And $\epsilon_{123} = -\epsilon_{12}{}^3 = 1, \ \eta_{ij} = -\delta_{ij}, \ i,j = 1,2,3.$

[2] In [22] and in literatures, both are called the para-Poincaré algebras.

algebras $\mathfrak{h}_\pm$ and other non-relativistic algebras as well as their second ones by means of the combinatory approach. In section IV, we illustrate the relations among these kinematics and some physical implications. Finally, we end with some remarks.

II. THE INERTIAL MOTION ALGEBRA, THE SPECIAL RELATIVITY TRIPLE AND ITS GEOMETRIC COUNTERPART

A. The principle of relativity with (c, l) and the Umov-Weyl-Fock-transformations

In the inertial coordinate frames $\mathcal{F} := \{S(x)\}$, a free particle takes inertial motion[3]

$$x^i = x_0^i + v^i(t - t_0), \quad v^i = \frac{dx^i}{dt} = \text{consts.} \quad i = 1, 2, 3. \tag{2}$$

What are the most general transformations $\mathcal{T} := \{T\}$

$$\mathcal{T} \ni T: \quad x'^\mu = f^\mu(x, T), \; x^0 = ct, \; \mu = 0, \cdots, 3, \tag{3}$$

that keep the inertial motion (2) invariant?

It can be proved [17–20]: *The most general form of the transformations (3) are the LFTs*

$$T: \quad l^{-1}x'^\mu = \frac{A^\mu{}_\nu l^{-1} x^\nu + b^\mu}{c_\lambda l^{-1} x^\lambda + d} \tag{4}$$

and

$$det\ T = \begin{vmatrix} A & b^t \\ c & d \end{vmatrix} = 1, \tag{5}$$

where $A = \{A^\mu{}_\nu\}$ a 4×4 matrix, b, c 1×4 matrixes, $d \in R$ and superscript ^{t} for the transpose.

Clearly, all Umov-Weyl-Fock transformations $T \in \mathcal{T}$ form the inertial motion transformation group $IM(1, 3)$ of twenty-four generators with its algebra $\mathfrak{im}(1, 3)$. Actually, the inertial motion (2) may be viewed as a straight line and the most general transformations among straight lines form the real projective group $RP(4)$ (see, e.g. [6, 7]). But, for keeping the orientation the antipodal identification should not be taken [8].

Further, all algebraic relations must keep the invariance of time reversal and space inversion.

B. Three kinds of special relativity as a triple and algebraic relations

As in [3], we first consider the Lorentz isotropy and the transitivity by the Euclid translations of Abelian group $R(1, 3)$ generated by H and $\mathbf{P}_i$ for coordinates afterwards. Then, it follows the Poincaré transformations of $ISO(1, 3) = R(1, 3) \rtimes SO(1, 3)$ and both isotropy and transitivity make

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[3] Conventionally, the concepts are depended on certain metric introduced. These had been generalized in [3, 18–20].

the Minkowski ($Mink$) spacetime as a 4d homogeneous space $M = ISO(1,3)/SO(1,3)$. The Poincaré algebra $\mathfrak{iso}(1,3)$, i.e. $\mathfrak{p}$, is generated by the set $\{T\}^{\mathfrak{p}} := (H, \mathbf{P}_i, \mathbf{K}_i, \mathbf{J}_i)$

$$\begin{aligned} H = \partial_t, \quad \mathbf{P}_i = \partial_i, \quad \mathbf{K}_i = t\partial_i - c^{-2}x_i\partial_t, \\ \mathbf{J}_i = \frac{1}{2}\epsilon_i{}^{jk}L_{jk}, \quad L_{jk} := x_j\partial_k - x_k\partial_j, \end{aligned} \tag{6}$$

where $x_\mu := \eta_{\mu\nu}x^\nu, \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and $\epsilon_i{}^{jk} = \epsilon_{ijk}$.

In order to get the dS/AdS transformations with Lorentz isotropy, the Euclid translations $(H, \mathbf{P}_i)$ should be replaced by the Beltrami translations $(H^\pm, \mathbf{P}_i^\pm)$ [3], respectively

$$H^\pm = \partial_t \mp \nu^2 t x^\nu \partial_\nu, \quad \mathbf{P}_i^\pm = \partial_i \mp l^{-2} x_i x^\nu \partial_\nu. \tag{7}$$

Thus, from the Beltrami model of dS/AdS-spacetime [3], it follows the LFTs, denoted by $\mathcal{S}_\pm$, of dS/AdS-group $SO(1,4)/SO(2,3) \subset IM_L(1,3)$, which keep the coordinate domain conditions and the Beltrami metrics invariant, respectively. Actually, the Beltrami-dS/AdS-spacetime $\mathcal{B}_\pm$ is also a homogeneous space $\mathcal{B}_\pm \cong \mathcal{S}_\pm/SO(1,3)$, respectively. Then the sets $\{T\}^{\mathfrak{d}_\pm} := (H^\pm, \mathbf{P}_i^\pm, \mathbf{K}_i, \mathbf{J}_i)$ span the dS/AdS algebra $\mathfrak{so}(1,4), \mathfrak{so}(2,3)$ or $\mathfrak{d}_\pm$, respectively. And the dS/AdS doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ follows.

Further, in the space of $\mathfrak{im}_L(1,3)$, the translations are related by

$$H = \frac{1}{2}(H^+ + H^-), \quad \mathbf{P}_i := \frac{1}{2}\left(\mathbf{P}_i^+ + \mathbf{P}_i^-\right), \tag{8}$$

where the constant l is hidden, while the dimensions of the generators and algebra are still kept.

In addition to all these generators of $\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{p}$ for three kinds of special relativity, there are still other ten generators in the $\mathfrak{im}_L(1,3)$ as follows

$$\mathbf{N}_i = t\partial_i + c^{-2}x_i\partial_t, \quad R_{ij} = R_{ji} = x_i\partial_j + x_j\partial_i, (i<j), \quad M_\mu = x^{(\mu)}\partial_{(\mu)}, \tag{9}$$

where no summation is taken for repeated indexes in brackets.

Together with the dS/AdS relations, the rest non-vanishing relations of $\mathfrak{im}_L(1,3)$ of the set $\{T\}^{\mathfrak{im}} := (H^\pm, \mathbf{P}_i^\pm, \mathbf{J}_i, \mathbf{K}_i, \mathbf{N}_i, M_0, M_i, R_{ij})$ read:

$$\begin{array}{ll} [\mathbf{P}_i^+, \mathbf{P}_j^-] = (1-\delta_{(i)(j)})l^{-2}R_{(i)(j)} - 2l^{-2}\delta_{i(j)}(M_{(j)} + \Sigma_\kappa M_\kappa), & \\ [\mathbf{P}_i^\pm, M_j] = \delta_{i(j)}\mathbf{P}_{(j)}^\mp, & [\mathbf{P}_i^\pm, R_{jk}] = -\delta_{ij}\mathbf{P}_k^\mp - \delta_{ik}\mathbf{P}_j^\mp, \\ [H^+, H^-] = 2\nu^2\left(M_0 + \Sigma_\kappa M_\kappa\right), & [H^\pm, M_0] = H^\mp, \\ [\mathbf{K}_i, M_0] = -\mathbf{N}_i, & [\mathbf{K}_i, M_j] = \delta_{i(j)}\mathbf{N}_{(j)}, \\ [\mathbf{K}_i, R_{jk}] = -\delta_{ij}\mathbf{N}_k - \delta_{ik}\mathbf{N}_j, & [\mathbf{N}_i, M_0] = -\mathbf{K}_i, \\ [\mathbf{N}_i, M_j] = \delta_{i(j)}\mathbf{K}_{(j)}, & [\mathbf{N}_i, R_{jk}] = -\delta_{ij}\mathbf{K}_k - \delta_{ik}\mathbf{K}_j, \\ [\mathbf{K}_i, \mathbf{N}_j] = (\delta_{(i)(j)} - 1)c^{-2}R_{(i)(j)} - 2\delta_{(i)j}c^{-2}\left(M_0 - M_{(i)}\right), & \\ [L_{ij}, M_k] = \delta_{j(k)}R_{i(k)} - \delta_{i(k)}R_{j(k)}, & [R_{ij}, M_k] = \delta_{i(k)}L_{j(k)} + \delta_{j(k)}L_{i(k)}, \\ [L_{ij}, R_{kl}] = 2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})(M_i - M_j) + \delta_{ik}R_{jl} + \delta_{il}R_{jk} - \delta_{jk}R_{il} - \delta_{jl}R_{ik}, & \\ [R_{ij}, R_{kl}] = -\delta_{ik}R_{jl} - \delta_{il}R_{jk} - \delta_{jk}R_{il} - \delta_{jl}R_{ik}. & \end{array} \tag{10}$$

The algebra $\mathfrak{im}_L(1,3)$ is the closed algebra for the doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$.

Clearly, in addition to the common Lorentz isotropy and the relation between the Euclid and Beltrami translations $\mathbf{P}_\mu, \mathbf{P}^{\pm}_\mu$, four generators (M_0, M_i) and six generators $(\mathbf{N}_i, R_{ij})$ of $\mathfrak{im}_L(1,3)$ exchange these translations from one to another. Thus, three kinds of special relativity act as a whole in $IM_L(1,3)$, called *the special relativity triple*[3].

There are rich algebraic relations in the triple. From Eq. (8), it follows naturally the pseudo-time/space-translations by linear combination

$$H' = \frac{1}{2}(H^+ - H^-) = -\nu^2 t x^\mu \partial_\mu, \quad \mathbf{P}'_i := \frac{1}{2}\left(\mathbf{P}^+_i - \mathbf{P}^-_i\right) = -l^{-2} x_i x^\nu \partial_\nu. \tag{11}$$

The set $\{T\}^{\mathfrak{p}_2} := (H', \mathbf{P}'_i, \mathbf{K}_i, \mathbf{J}_i)$ spans the second Poincaré algebra $\mathfrak{p}_2$. It is important that this algebra preserves the $Mink$-light cone at the origin.

Thus, dual to the dS/AdS doublet $(\mathfrak{d}_+, \mathfrak{d}_-)$ there is the Poincaré doublet $(\mathfrak{p}, \mathfrak{p}_2)$ and both can be transformed from one to another by either the linear combinations of the generators or the algebraic relation in $\mathfrak{im}(1,3)$. In fact, with respect to the $\mathfrak{im}(1,3)$, there is a $dS/AdS/$Poincaré algebraic quadruplet $(\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{p}, \mathfrak{p}_2)$ made simplicially of 4 algebraic triplets $(\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{p})$, $(\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{p}_2)$, $(\mathfrak{d}_+, \mathfrak{p}, \mathfrak{p}_2)$ and $(\mathfrak{d}_-, \mathfrak{p}, \mathfrak{p}_2)$, as well as 6 doublets $(\mathfrak{d}_+, \mathfrak{d}_-), (\mathfrak{p}, \mathfrak{p}_2)$, $(\mathfrak{d}_+, \mathfrak{p}), (\mathfrak{d}_+, \mathfrak{p}_2)$, $(\mathfrak{d}_-, \mathfrak{p})$ and $(\mathfrak{d}_-, \mathfrak{p}_2)$. And the closed algebra of the quadruplet is also the $\mathfrak{im}(1,3)$.

C. The geometry triple with $\mathfrak{so}(4)$ isotropy and algebraic relations

If we replace the Lorentz boost $\mathbf{K}_i$ by the geometric boost $\mathbf{N}_i$ and at the same time, due to the signature -2 of $\eta_{\mu\nu}$ we adjust the Beltrami time and space translations suitably, the relativistic algebras turn to their geometric counterparts. That is, the Riemann/Lobachevsky/Euclid algebras $\mathfrak{so}(5), \mathfrak{so}(4,1), \mathfrak{iso}(4)$ or $\mathfrak{r}, \mathfrak{l}, \mathfrak{e}$ with generator sets $\{T\}^{\mathfrak{r}} := (H^-, \mathbf{P}^+_i, \mathbf{N}_i, \mathbf{J}_i)$, $\{T\}^{\mathfrak{l}} := (H^+, \mathbf{P}^-_i, \mathbf{N}_i, \mathbf{J}_i)$ and $\{T\}^{\mathfrak{e}} := (H, \mathbf{P}_i, \mathbf{N}_i, \mathbf{J}_i)$ for 4d Riemann sphere, Lobachevsky hyperboloid and Euclid space, respectively. Thus, there is a geometry triple with rich algebraic relations.

First, there is also the second Euclid algebra $\mathfrak{e}_2$ with generator set $\{T\}^{\mathfrak{e}_2} := (H', \mathbf{P}'_i, \mathbf{N}_i, \mathbf{J}_i)$. Then, corresponding to the relativistic quadruplet, there is a geometric algebraic quadruplet $(\mathfrak{r}, \mathfrak{l}, \mathfrak{e}, \mathfrak{e}_2)$, too.

It should be noticed that the geometry triple is in the sense of the straight-lines as the geometric 'inertial motions' in the inertial frames with one geometric 'time' coordinate t and three space coordinates x^i. Although two universal constants (c, l) are still there, the signatures of the metrics are intrinsically changed to the geometric one, respectively.

III. THE NON-RELATIVISTIC KINEMATICS AND ALGEBRAIC RELATIONS

A. The Euclid time and space translation algebras: Galilei, Carroll and counterparts

The Galilei transformations as a subset of Umov-Weyl-Fock transformations (3) can be easily reached from the space isotropy generated by $\mathbf{J}_i$ with the Galilei boost $\mathbf{K}^{\mathfrak{g}}_i$ at origin as well as by the time and space translations $H, \mathbf{P}_i$ for transitivity. The generators can be easily obtained

$$H := \partial_t = \frac{1}{2}\left(H^+ + H^-\right), \quad \mathbf{P}_i := \partial_i = \frac{1}{2}\left(\mathbf{P}^+_i + \mathbf{P}^-_i\right), \quad \mathbf{K}^{\mathfrak{g}}_i := t\partial_i = \frac{1}{2}\left(\mathbf{K}_i + \mathbf{N}_i\right). \tag{12}$$

TABLE I: All possible relativistic, geometric and non-relativistic kinematic algebras

Algebra	Symbol	Generator set[a]	$[\mathcal{H},\mathbf{P}]$	$[\mathcal{H},\mathbf{K}]$	$[\mathbf{P},\mathbf{P}]$	$[\mathbf{K},\mathbf{K}]$	$[\mathbf{P},\mathbf{K}]$
dS	$\mathfrak{d}_+$	$(H^+,\mathbf{P}^+,\mathbf{K},\mathbf{J})$	$\nu^2\mathbf{K}$	$\mathbf{P}$	$l^{-2}\mathbf{J}$	$-c^{-2}\mathbf{J}$	$c^{-2}\mathcal{H}$
AdS	$\mathfrak{d}_-$	$(H^-,\mathbf{P}^-,\mathbf{K},\mathbf{J})$	$-\nu^2\mathbf{K}$	$\mathbf{P}$	$-l^{-2}\mathbf{J}$	$-c^{-2}\mathbf{J}$	$c^{-2}\mathcal{H}$
Poincaré	$\mathfrak{p}$ $\mathfrak{p}_2$	$(H,\mathbf{P},\mathbf{K},\mathbf{J})$ $\{H',\mathbf{P}',\mathbf{K},\mathbf{J}\}$	0	$\mathbf{P}$	0	$-c^{-2}\mathbf{J}$	$c^{-2}\mathcal{H}$
Riemann	$\mathfrak{r}$	$(H^-,\mathbf{P}^+,\mathbf{N},\mathbf{J})$	$-\nu^2\mathbf{K}$	$\mathbf{P}$	$l^{-2}\mathbf{J}$	$c^{-2}\mathbf{J}$	$-c^{-2}\mathcal{H}$
Lobachevsky	$\mathfrak{l}$	$(H^+,\mathbf{P}^-,\mathbf{N},\mathbf{J})$	$\nu^2\mathbf{K}$	$\mathbf{P}$	$-l^{-2}\mathbf{J}$	$c^{-2}\mathbf{J}$	$-c^{-2}\mathcal{H}$
Euclid	$\mathfrak{e}$ $\mathfrak{e}_2$	$(H,\mathbf{P},\mathbf{N},\mathbf{J})$ $(H',\mathbf{P}',\mathbf{N},\mathbf{J})$	0	$\mathbf{P}$	0	$c^{-2}\mathbf{J}$	$-c^{-2}\mathcal{H}$
Galilei	$\mathfrak{g}$ $\mathfrak{g}_2$	$(H,\mathbf{P},\mathbf{K}^{\mathfrak{g}},\mathbf{J})$ $(H',\mathbf{P}',\mathbf{K}^{\mathfrak{c}},\mathbf{J})$	0	$\mathbf{P}$	0	0	0
Carroll	$\mathfrak{c}$ $\mathfrak{c}_2$	$(H,\mathbf{P},\mathbf{K}^{\mathfrak{c}},\mathbf{J})$ $(H',\mathbf{P}',\mathbf{K}^{\mathfrak{g}},\mathbf{J})$	0	0	0	0	$c^{-2}\mathcal{H}$
NH_+	$\mathfrak{n}_+$ $\mathfrak{n}_{+2}$	$(H^+,\mathbf{P},\mathbf{K}^{\mathfrak{g}},\mathbf{J})$ $(H^+,\mathbf{P}',\mathbf{K}^{\mathfrak{c}},\mathbf{J})$	$\nu^2\mathbf{K}$	$\mathbf{P}$	0	0	0
NH_-	$\mathfrak{n}_-$ $\mathfrak{n}_{-2}$	$(H^-,\mathbf{P},\mathbf{K}^{\mathfrak{g}},\mathbf{J})$ $(-H^-,\mathbf{P}',\mathbf{K}^{\mathfrak{c}},\mathbf{J})$	$-\nu^2\mathbf{K}$	$\mathbf{P}$	0	0	0
para-Galilei	$\mathfrak{g}'$ $\mathfrak{g}'_2$	$(H',\mathbf{P},\mathbf{K}^{\mathfrak{g}},\mathbf{J})$ $(H,\mathbf{P}',\mathbf{K}^{\mathfrak{c}},\mathbf{J})$	$\nu^2\mathbf{K}$	0	0	0	0
HN_+	$\mathfrak{h}_+$ $\mathfrak{h}_{+2}$	$(H,\mathbf{P}^+,\mathbf{K}^{\mathfrak{c}},\mathbf{J})$ $(H',\mathbf{P}^+,\mathbf{K}^{\mathfrak{g}},\mathbf{J})$	$\nu^2\mathbf{K}$	0	$l^{-2}\mathbf{J}$	0	$c^{-2}\mathcal{H}$
HN_-	$\mathfrak{h}_-$ $\mathfrak{h}_{-2}$	$(H,\mathbf{P}^-,\mathbf{K}^{\mathfrak{c}},\mathbf{J})$ $(-H',\mathbf{P}^-,\mathbf{K}^{\mathfrak{g}},\mathbf{J})$	$-\nu^2\mathbf{K}$	0	$-l^{-2}\mathbf{J}$	0	$c^{-2}\mathcal{H}$
Static	$\mathfrak{s}$ $\mathfrak{s}_2$	$(H^{\mathfrak{s}},\mathbf{P}',\mathbf{K}^{\mathfrak{c}},\mathbf{J})$[b] $(H^{\mathfrak{s}},\mathbf{P},\mathbf{K}^{\mathfrak{g}},\mathbf{J})$	0	0	0	0	0

[a]All generators and commutators have right dimensions expressed by the universal constants c, l or ν.
[b]The generator $H^{\mathfrak{s}}$ is meaningful only when the central extension is considered.

Then the set $\{T\}^{\mathfrak{g}} := (H, \mathbf{P}_i, \mathbf{K}_i^{\mathfrak{g}}, \mathbf{J}_i)$ spans the Galilei algebra $\mathfrak{g}$.

From the viewpoint of combinatory, dual to the Galilei boost $\mathbf{K}_i^{\mathfrak{g}}$, there is the Carroll boost

$$\mathbf{K}_i^{\mathfrak{c}} := \frac{1}{2}(\mathbf{K}_i - \mathbf{N}_i) = -c^{-2} x_i \partial_t. \tag{13}$$

Actually, the set $\{T\}^{\mathfrak{c}} := (H, \mathbf{P}_i, \mathbf{K}_i^{\mathfrak{c}}, \mathbf{J}_i)$ spans the Carroll algebra $\mathfrak{c} \subset \mathfrak{im}(1,3)$. Thus, the Carroll kinematics is also based on the $PoR_{c,l}$. Clearly, there is a Galilei-Carroll doublet $(\mathfrak{g}, \mathfrak{c})$. And, the other two sets $\{T\}^{\mathfrak{g}_2} := (H', \mathbf{P}'_i, \mathbf{K}_i^{\mathfrak{c}}, \mathbf{J}_i)$ and $\{T\}^{\mathfrak{c}_2} := (H', \mathbf{P}'_i, \mathbf{K}_i^{\mathfrak{g}}, \mathbf{J}_i)$ with pseudo-translations span the second Galilei and Carroll algebras $\mathfrak{g}_2$ and $\mathfrak{c}_2$, respectively.

Thus, there are three more Galilei-Carroll doublets $(\mathfrak{c}, \mathfrak{g}_2)$, $(\mathfrak{g}, \mathfrak{c}_2)$ and $(\mathfrak{c}_2, \mathfrak{g}_2)$ as well as the Galilei or Carroll doublets $(\mathfrak{g}, \mathfrak{g}_2)$ or $(\mathfrak{c}, \mathfrak{c}_2)$, respectively. In fact, there is the Galilei-Carroll algebraic quadruplet $(\mathfrak{g}, \mathfrak{g}_2, \mathfrak{c}, \mathfrak{c}_2)$ made of 4 triplets and 6 doublets simplicially with respect to the $\mathfrak{im}(1,3)$.

B. The Beltrami time translation algebras: Newton-Hooke/anti-Newton-Hooke/para-Galilei and counterparts

If we replace the time translation by Beltrami time translations $H^{\pm}$ in (7), the sets $\{T\}^{\mathfrak{n}_\pm} := (H^{\pm}, \mathbf{P}_j, \mathbf{K}_j^{\mathfrak{g}}, \mathbf{J}_j)$ span the $NH_\pm$ algebra $\mathfrak{n}_\pm$, respectively. And the other two sets $\{T\}^{\mathfrak{n}_{\pm 2}} := (\pm H^{\pm}, \mathbf{P}'_i, \mathbf{K}_i^{\mathfrak{c}}, \mathbf{J}_i)$ span the second $NH_\pm$ algebras $\mathfrak{n}_{\pm 2}$. Hence, there is the $NH_\pm$ quadruplet $(\mathfrak{n}_\pm, \mathfrak{n}_{\pm 2})$ made of four triplets and six doublets simplicially.

Furthermore, again with the pseudo-translation $H', \mathbf{P}'_i$ in (11) the sets $\{T\}^{\mathfrak{g}'} := (H', \mathbf{P}_j, \mathbf{K}^{\mathfrak{g}}_j, \mathbf{J}_j)$ and $\{T\}^{\mathfrak{g}'_2} := (H, \mathbf{P}'_i, \mathbf{K}^{\mathfrak{c}}_i, \mathbf{J}_i)$ span the para-Galilei algebra $\mathfrak{g}'$ and the second one $\mathfrak{g}'_2$, respectively. Then, the para-Galilei doublet $(\mathfrak{g}', \mathfrak{g}'_2)$ follows.

Thus, the $NH_\pm$-para-Galilei sextuplet $(\mathfrak{n}_\pm, \mathfrak{n}_{\pm 2}, \mathfrak{g}', \mathfrak{g}'_2)$ can be made of simplicialy.

C. The Beltrami space translation algebras: Hooke-Newton/anti-Hooke-Newton/static and counterparts

If we replace the space translations $\mathbf{P}_i$ by the Beltrami translations $\mathbf{P}^\pm_i$ in (7), still keep the time translation H, two sets $\{T\}^{\mathfrak{h}_\pm} := (H, \mathbf{P}^\pm, \mathbf{K}^{\mathfrak{c}}_i, \mathbf{J}_i)$ span the $HN_\pm$ algebra $\mathfrak{h}_\pm \cong \mathfrak{iso}(4)/\mathfrak{iso}(1,3)$, respectively. And the $HN_\pm$ doublet $(\mathfrak{h}_+, \mathfrak{h}_-)$ follows.

From algebraic relations of $\mathfrak{h}_\pm$, it follows that the sub-sets $(\mathbf{P}^\pm_i, \mathbf{J}_i)$ form subalgebras $\mathfrak{so}(4), \mathfrak{so}(1,3)$, which keeps a 3d Riemann sphere/Lobachevsky hyperploid S^3/H^3, respectively, and others form 4d Abelian subalgebras. Since the time and space are also separated in analogy to the cases of the $NH_\pm$, it would be better to name them $HN_\pm$, rather than para-Poincaré as was named in [22]. In addition, exchanging the generators $(H^\pm, \mathbf{P}_i, \mathbf{K}^{\mathfrak{g}}_i) \subset \{T\}^{\mathfrak{n}_\pm}$ in the $NH_\pm$ algebras by the generators $(H, \mathbf{P}^\pm_i, \mathbf{K}^{\mathfrak{c}}_i) \subset \{T\}^{\mathfrak{h}_\pm}$, the $HN_\pm$ algebras follow and vice versa. This is also why the algebras are called after the name Hooke-Newton, rather than para-Poincaré.

It can also be shown that the sets $\{T\}^{\mathfrak{h}_{\pm 2}} := (\pm H', \mathbf{P}^\pm_i, \mathbf{K}^{\mathfrak{g}}_i, \mathbf{J}_i)$ span the second $HN_\pm$ algebras $\mathfrak{h}_{\pm 2} \cong \mathfrak{iso}(4)/\mathfrak{iso}(1,3)$, respectively. Thus, not only more $HN_\pm$ doublets follow, but also the $HN_\pm$ quadruplet $(\mathfrak{h}_\pm, \mathfrak{h}_{\pm 2})$ made of simplicially follows.

In the combinatory approach, the so-called static algebra[22] can also be reached with

$$H^{\mathfrak{s}} = H - H = H' - H' = 0 \tag{14}$$

by combining the generator sets of $HN_\pm$ or $HN_{\pm 2}$ algebras. In other words, two sets $\{T\}^{\mathfrak{s}} := (H^{\mathfrak{s}}, \mathbf{P}'_i, \mathbf{K}^{\mathfrak{c}}_i, \mathbf{J}_i)$ and $\{T\}^{\mathfrak{s}_2} := (H^{\mathfrak{s}}, \mathbf{P}_i, \mathbf{K}^{\mathfrak{g}}_i, \mathbf{J}_i)$ span two static algebras $\mathfrak{s}$ and $\mathfrak{s}_2$, respectively, with a doublet $(\mathfrak{s}, \mathfrak{s}_2)$. In fact, $H^{\mathfrak{s}} = 0$ is not a generator. If the central extension is considered, $H^{\mathfrak{s}}$ becomes the one in [22].

It is clear that there also exist the $HN_\pm$-static algebraic sextuplet $(\mathfrak{h}_\pm, \mathfrak{h}_{\pm 2}, \mathfrak{s}, \mathfrak{s}_2)$ made of the triplet $(\mathfrak{h}_+, \mathfrak{h}_-, \mathfrak{s})$ and so on simplicially.

IV. THE RELATIONS AMONG KINEMATICS AND SOME IMPLICATIONS

In the combinatory approach, all these kinematical algebras are related in the space of $\mathfrak{im}(1,3)$ either by suitably exchanging some generators or by their algebraic relations with the generators (M_0, M_i) and R_{ij}, no matter whether the algebras are relativistic, geometric or non-relativistic. Let us illustrate some of these relations and relevant physical implications.

As was mentioned, there is a duality between the relativistic and geometric kinematics and their algebraic relations rather than the Wick rotation [10]. Actually, we may also consider a relativistic-geometric octuplet with all eight algebras as singlets simplicially and so on.

There is the Poincaré-Euclid-Galilei-Carroll quadruple $(\mathfrak{p}, \mathfrak{e}, \mathfrak{g}, \mathfrak{c})$, as well as its mimic of the second copies, is a typical relativistic-geometric-non-relativistic one, in which there is the Poincaré-Euclid doublet $(\mathfrak{p}, \mathfrak{e})$ that implicates the relation between the $Mink$-spacetime and the 4d Euclid space different from the Wick rotation[10]. And the Pincaré-Galilei-Carroll triplet $(\mathfrak{p}, \mathfrak{g}, \mathfrak{c})$ is a typical relativistic-non-relativistic one. As for non-relativistic kinematics, the Galilei and Carroll algebras, which form a doublet $(\mathfrak{g}, \mathfrak{c})$, indicate that the Newtonian kinematics is for the motions of $v \ll c$, while the Carroll kinematics may be for a kind of extremely superluminary motions with $v \gg c$, having invariant constant c. Since the algebraic combination of the Galilei boost $\mathbf{K}_i^{\mathfrak{g}}$ and the Carroll one $\mathbf{K}_i^{\mathfrak{c}}$ leads to the Lorentz boost $\mathbf{K}_i$, it should be for a relativistic one. This is just the case for the Poincaré kinematics. Taking into account the dS-Poincaré triplet $(\mathfrak{d}_+, \mathfrak{p}, \mathfrak{p}_2)$, the Galilei kinematics can be related to the dS one and vice versa.

Similarly, in the dS/AdS-$NH_\pm$ quadruple $(\mathfrak{d}_+, \mathfrak{d}_-, \mathfrak{n}_\pm)$, there are the dS/AdS-$NH_\pm$ triplets $(\mathfrak{d}_+, \mathfrak{n}_\pm)$ and $(\mathfrak{d}_-, \mathfrak{n}_\pm)$ together with other two triplets, which also make sense, respectively.

As non-relativistic algebras, the Galilei $\mathfrak{g}$ and $NH_\pm$ $\mathfrak{n}_\pm$ share nine common generators $(\mathbf{P}_j, \mathbf{K}_j^{\mathfrak{g}}, \mathbf{J}_j)$. In the Galilei-$NH_\pm$ quadruplet $(\mathfrak{g}, \mathfrak{g}_2, \mathfrak{n}_\pm)$, the triplet $(\mathfrak{g}, \mathfrak{n}_\pm)$ made of three doublets $(\mathfrak{g}, \mathfrak{n}_+)$, $(\mathfrak{g}, \mathfrak{n}_-)$ and $(\mathfrak{n}_+, \mathfrak{n}_-)$ and others also indicate certain sense in physics. In general, for Galilei-$NH_\pm$ there is a sextuplet $(\mathfrak{g}, \mathfrak{g}_2, \mathfrak{n}_\pm, \mathfrak{n}_{\pm 2})$ made of simplicially by lower Galilei-$NH_\pm$ multiplets.

There are also the Galilei-para-Galilei-$NH_\pm$ quadruplet $(\mathfrak{g}, \mathfrak{g}', \mathfrak{n}_\pm)$ made of lower multiplets with common generators $(\mathbf{J}, \mathbf{P}, \mathbf{K})$. Similarly, their partners form another quadruplet $(\mathfrak{g}_2,\ \mathfrak{g}_2',\ \mathfrak{n}_{\pm 2})$. Combining them together simplicially, there is the Galilei-para-Galilei-$NH_\pm$ octuplet and so on.

For the $HN_\pm$ algebras $(\mathfrak{h}_\pm, \mathfrak{h}_{\pm 2})$ and the static algebras $(\mathfrak{s}, \mathfrak{s}_2)$, they can also be linked with other algebras within certain multiplets simplicially.

In fact, there are fourteen-type kinematic algebras[4] with twenty-four copies in the space of $\mathfrak{im}(1,3)$. Thus, there is a twenty-four-plet for all them made of lower multiplets simplicially.

V. CONCLUDING REMARKS

We have shown that all possible kinematics can be set up based on the $PoR_{c,l}$ and all these kinematic algebras with ten generators are related. Thus, their physical implications should be explored based on the $PoR_{c,l}$. Since the dS kinematics is very important for the cosmic scale physics characterized by the cosmological constant [8, 15, 16], all other related kinematics may have certain physical meaning in their own right.

Different from traditional considerations, two universal constants c and l always exist there, which make sense at least for right dimensions of coordinates, generators and algebraic relations.

Those algebras with the pseudo-translations H' and $\mathbf{P}_i'$ cannot make transitivity in time and space and might be ignored in space-time physics. However, not only they do make sense in combinatory, but also still play some physical roles. For example, the second Poincaré algebra $\mathfrak{p}_2$ preserves the $Mink$-light cone, then with the translations $\mathbf{P}_j$ and others span the whole algebra $\mathfrak{im}(1,3)$. Moreover,

[4] The static algebra is a special one.

by combining the Galilei and Carroll boosts $\mathbf{K}_i^{\mathfrak{g}/\mathfrak{c}}$, it follows the Poincaré algebra $\mathfrak{p}$, which is related to the dS algebra $\mathfrak{d}_+$ by the combination of its translations and pseudo-translations of $\mathfrak{p}_2$.

The sign of the pseudo-translations H', $\mathbf{P}'_i$ and the Carroll boosts are determined by the algebraic relations. Since the correct dimensions must be kept for all generators and algebraic relations, once the sign of a generator is changed, the sign(s) of relevant generators must also be changed so that the original algebraic relations are still there as time reversal or space inversion are made.

In this paper, the order of the time and space coordinates are fixed. If it is changed, similar algebraic relations must occur. In some cases this makes sense. For instance, the kinematic algebras $\mathfrak{so}(2,3)$ with different orders can be regarded as the conformal extensions of relativistic algebras on the space of one dimension lower and so on [13].

The combinatory approach is very different from the contraction approach [21, 22]. Although we have mainly focused on the Lie algebraic aspects of kinematics, the results already indicate that this is also the case for corresponding groups, geometries with metrics and global aspects.

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